ECS 203 - Part 2A - For CPE2

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CHAPTER 6

Energy Storage Elements: Capacitors and Inductors

To this point in our study of electronic circuits, time has not been important. The analysis and designs we have performed so far have been static, and all circuit responses at a given time have depended only on the circuit inputs at that time. In this chapter, we shall introduce two important passive circuit elements: the capacitor and the inductor.

6.1. Introduction and a Mathematical Fact

6.1.1. Capacitors and inductors, which are the electric and magnetic duals of each other, differ from resistors in several significant ways.

- Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. They are called **storage elements**.
- Furthermore, their branch variables do not depend algebraically upon each other. Rather, their relations involve temporal derivatives and integrals. Thus, the analysis of circuits containing capacitors and inductors involve **differential equations** in time.

6.1.2. An important mathematical fact: Given

$$\frac{d}{dt}f(t) = g(t),$$

6.2. Capacitors

6.2.1. A capacitor is a passive element designed to store energy in its electric field. The word capacitor is derived from this element's capacity to store energy.

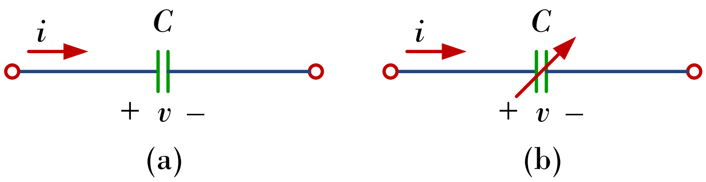
6.2.2. When a voltage source $v(t)$ is connected across the capacitor, the amount of charge stored, represented by q, is directly proportional to $v(t)$, i.e.,

$$q(t) = Cv(t)$$

where C, the constant of proportionality, is known as the **capacitance** of the capacitor.

- The unit of capacitance is the **farad (F)** in honor of Michael Faraday.
- 1 farad = 1 coulomb/volt.

6.2.3. Circuit symbol for capacitor of C farads:



6.2.4. Since $i = \frac{dq}{dt}$, then the current-voltage relationship of the capacitor is

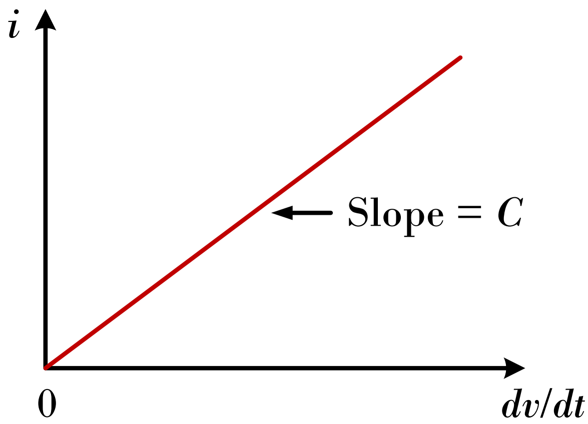
$$i = C\,\frac{dv}{dt}. \tag{6.2}$$

Note that in (6.2), the capacitance value C is constant (time-invariant) and that the current i and voltage v are both functions of time (time-varying). So, in fact, the full form of (6.2) is

$$i(t) = C\frac{d}{dt}v(t).$$

Hence, the voltage-current relation is

$$\boxed{v(t) = \frac{1}{C}\int_{t_o}^{t} i(\tau)d\tau + v(t_o)}$$



where $v(t_o)$ is the voltage across the capacitor at time t_o. Note that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has ***memory***.

6.2.5. The **instantaneous power** delivered to the capacitor is

$$p(t) = i(t) \times v(t) = \left(C \frac{d}{dt} v(t) \right) v(t).$$

The **energy** stored in the capacitor is

$$w(t) = \int_{-\infty}^{t} p(\tau) d\tau = \frac{1}{2} C v^2(t).$$

In the above calculation, we assume $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$.

6.2.6. Typical values

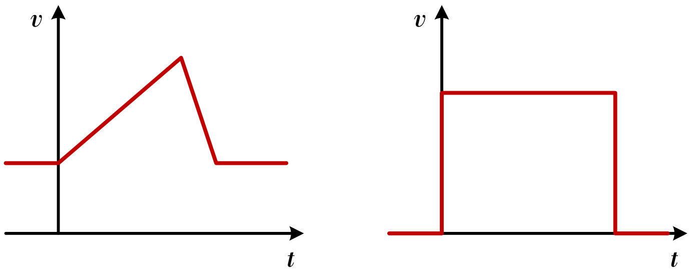
(a) Capacitors are commercially available in different values and types.
(b) Typically, capacitors have values in the picofarad (pF) to microfarad (μF) range.
(c) For comparison, two pieces of insulated wire about an inch long, when twisted together, will have a capacitance of about 1 pF.

6.2.7. Two important implications of (6.2):

(a) A capacitor is an open circuit to dc.

When the voltage across a capacitor is not changing with time (i.e., dc voltage), its derivative wrt. time is $\frac{dv}{dt} = 0$ and hence the current through the capacitor is $i(t) = C\frac{dv}{dt} = C \times 0 = 0$.

(b) The voltage across a capacitor cannot jump (change abruptly)
Because $i = C\,\frac{dv}{dt}$, a discontinuous change in voltage requires an infinite current, which is physically impossible.



6.2.8. Remark: An ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.

EXAMPLE 6.2.9. If a 10 μF is connected to a voltage source with

$$v(t) = 50 \sin 2000t \quad \text{V}$$

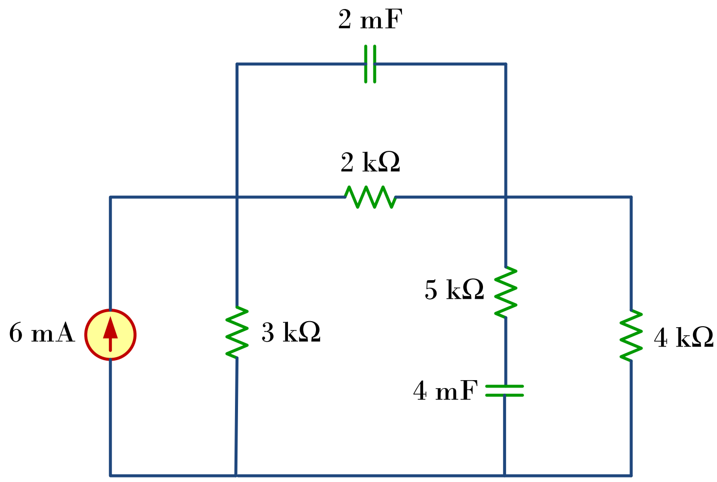
determine the current through the capacitor.

EXAMPLE 6.2.10. Determine the voltage across a 2-μF capacitor if the current through it is

$$i(t) = 6e^{-3000t} \quad \text{mA}$$

Assume that the initial capacitor voltage (at time $t = 0$) is zero.

EXAMPLE 6.2.11. Obtain the energy stored in each capacitor in the figure below under dc conditions.

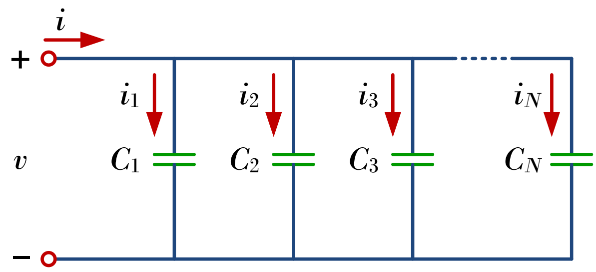


6.3. Series and Parallel Capacitors

We know from resistive circuits that series-parallel combination is a powerful tool for simplifying circuits. This technique can be extended to series-parallel connections of capacitors, which are sometimes encountered. We desire to replace these capacitors by a single equivalent capacitor C_{eq}.

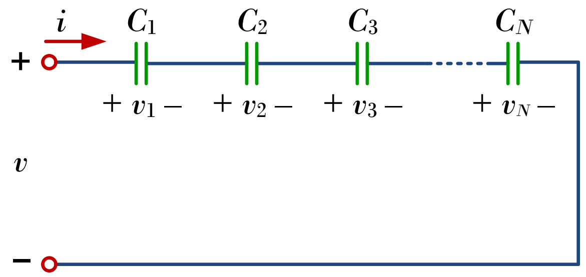
6.3.1. The equivalent capacitance of N **parallel**-connected capacitors is the sum of the individual capacitance.

$$C_{eq} = C_1 + C_2 + \cdots + C_N$$

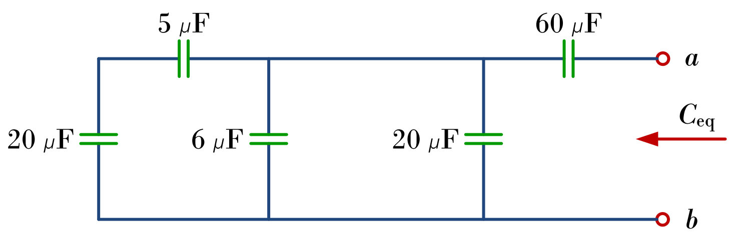


The equivalent capacitance of N **series**-connected capacitors is the the reciprocal of the sum of the reciprocals of the individual capacitances.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$$



Example 6.3.2. Find the C_{eq}.

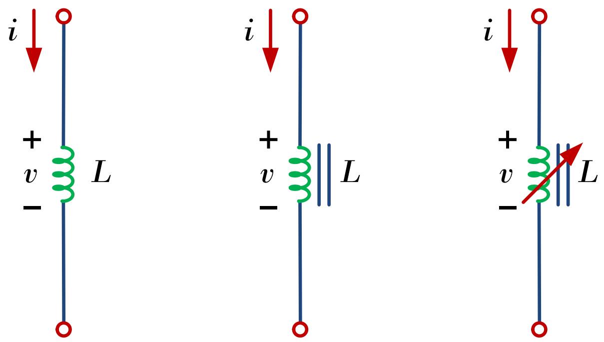


6.4. Inductors

6.4.1. An inductor is a passive element designed to store energy in its magnetic field.

6.4.2. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors.

6.4.3. Circuit symbol of inductor:



6.4.4. If a current is allowed to pass through an inductor, the voltage across the inductor is directly proportional to the time rate of change of the current, i.e.,

$$v(t) = L\,\frac{d}{dt}i(t), \tag{6.3}$$

where L is the constant of proportionality called the **inductance** of the inductor. The unit of inductance is **henry** (H), named in honor of Joseph Henry.

- 1 henry equals 1 volt-second per ampere.

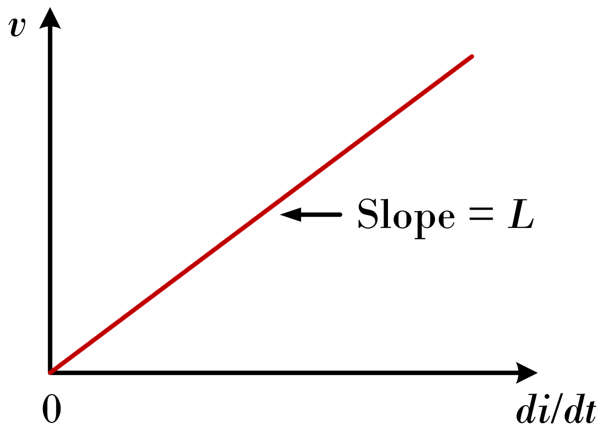
6.4.5. By integration, the current-voltage relation is

$$i(t) = \frac{1}{L}\int_{t_o}^{t} v(\tau)\ d\tau + i(t_o),$$

where $i(t_o)$ is the current at time t_o.

6.4.6. The instantaneous **power** delivered to the inductor is

$$p(t) = v(t) \times i(t) = \left(L\frac{d}{dt}i(t)\right) i(t)$$



The energy stored in the inductor is

$$w(t) = \int_{-\infty}^{t} p(\tau)\, d\tau = \frac{1}{2} L i^2(t).$$

6.4.7. Like capacitors, commercially available inductors come in different values and types. Typical practical inductors have inductance values ranging from a few microhenrys (μH), as in communication systems, to tens of henrys (H) as in power systems.

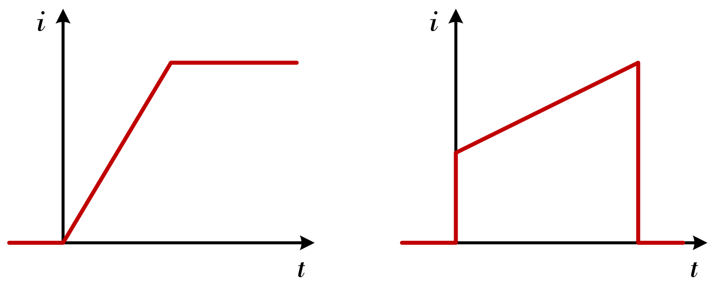
6.4.8. Two important implications of (6.3):

(a) An inductor acts like a short circuit to dc.

When the current through an inductor is not changing with time (i.e., dc current), its derivative wrt. time is $\frac{di}{dt} = 0$ and hence the voltage across the inductor is $v(t) = L\frac{di}{dt} = L \times 0 = 0$.

(b) The current through an inductor cannot change instantaneously.

This opposition to the change in current is an important property of the inductor. A discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible.



6.4.9. Remark: The ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.

EXAMPLE 6.4.10. If the current through a 1-mH inductor is $i(t) = 20\cos 100t$ mA, find the terminal voltage and the energy stored.

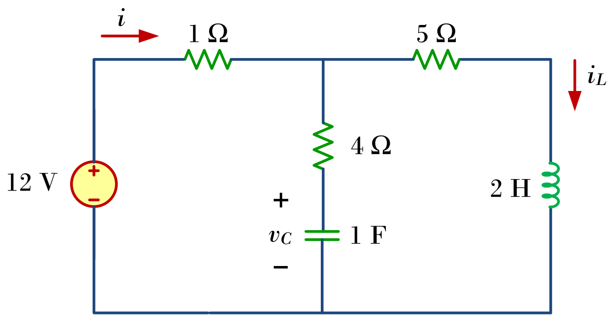
EXAMPLE 6.4.11. Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}.$$

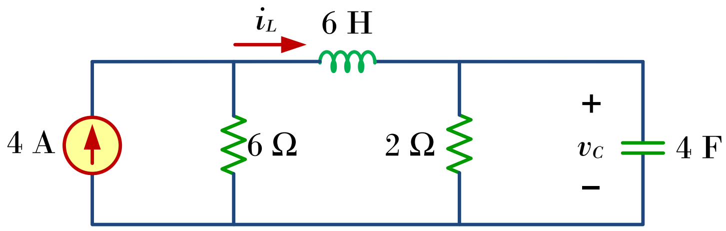
In addition, find the energy stored within $0 < t < 5$ s.

EXAMPLE 6.4.12. The terminal voltage of a 2-H inductor is $v(t) = 10(1 - t)$ V. Find the current flowing through it at $t = 4$ s and the energy stored in it within $0 < t < 4$ s. Assume $i(0) = 2$ A.

EXAMPLE 6.4.13. Determine v_C, i_L and the energy stored in the capacitor and inductor in the following circuit under dc conditions.



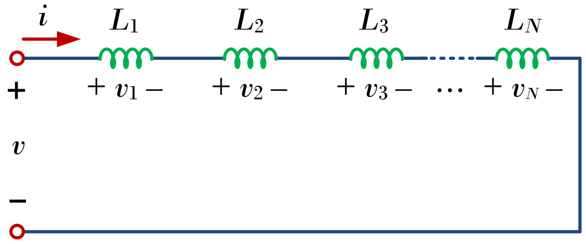
EXAMPLE 6.4.14. Determine v_C, i_L and the energy stored in the capacitor and inductor in the following circuit under dc conditions.



6.5. Series and Parallel Inductors

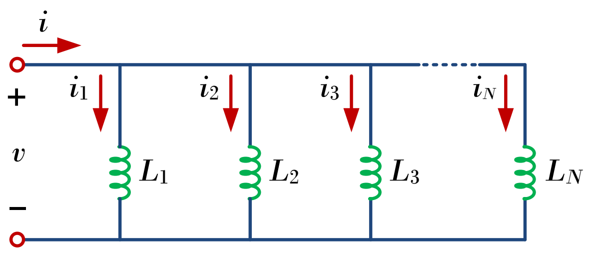
6.5.1. The equivalent inductance of N series-connected inductors is the sum of the individual inductances, i.e.,

$$L_{eq} = L_1 + L_2 + \cdots + L_N$$



6.5.2. The equivalent inductance of N parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances, i.e.,

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$



6.5.3. Remark: Note that

(a) inductors in series are combined in exactly the same way as resistors in series and

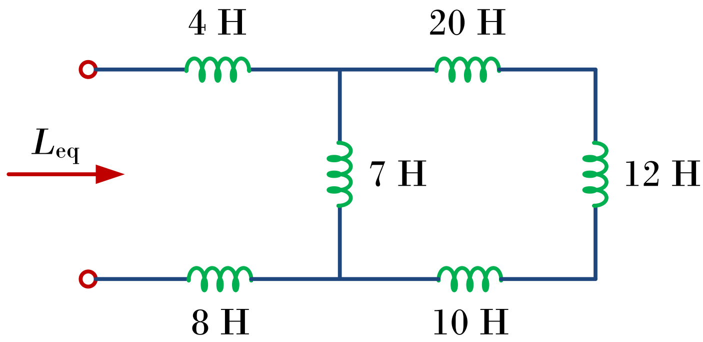
(b) inductors in parallel are combined in the same way as resistors in parallel.

Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

† *Passive sign convention is assumed.*

EXAMPLE 6.5.4. Find the equivalent inductance L_{eq} of the circuit shown below.



6.6. Applications: Integrators and Differentiators

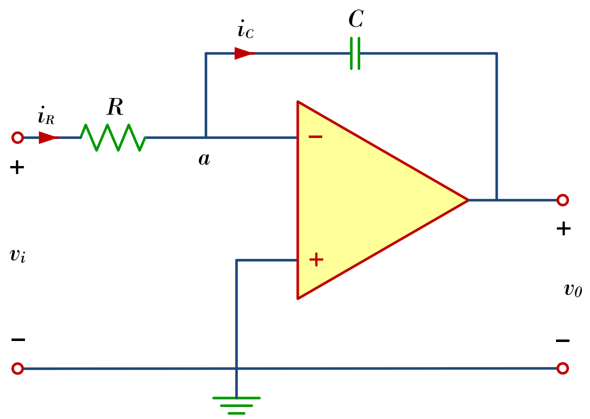
6.6.1. Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:

(a) The capacity to store energy makes them useful as temporary voltage or current sources. Thus, they can be used for generating a large amount of current or voltage for a short period of time.
(b) Capacitors oppose any abrupt change in voltage, while inductors oppose any abrupt change in current. This property makes inductors useful for spark or arc suppression and for converting pulsating dc voltage into relatively smooth dc voltage.
(c) Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination.

The first two properties are put to use in dc circuits, while the third one is taken advantage of in ac circuits.

In this final part of the chapter, we will consider two applications involving capacitors and op amps: integrator and differentiator.

6.6.2. An **integrator** is an op amp circuit whose output is proportional to the integral of the input signal. We obtain an integrator by replacing the feedback resistor R_f in the inverting amplifier by a capacitor.



This gives

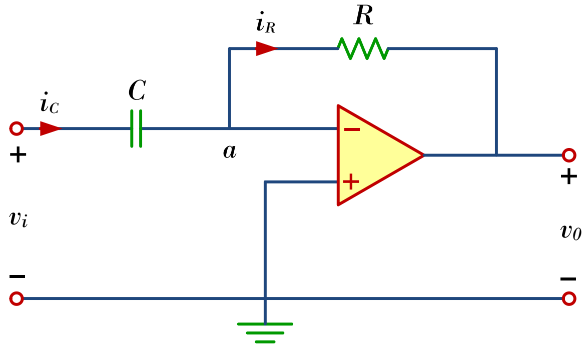
$$\frac{d}{dt}v_o(t) = -\frac{1}{RC}v_i(t),$$

which implies

$$v_o(t) = -\frac{1}{RC}\int_0^t v_i(\tau)d\tau + v_o(0).$$

- To ensure that $v_o(0) = 0$, it is always necessary to discharge the integrators capacitor prior to the application of a signal.
- In practice, the op amp integrator requires a feedback resistor to reduce dc gain and prevent saturation. Care must be taken that the op amp operates within the linear range so that it does not saturate.

6.6.3. A **differentiator** is an op amp circuit whose output is proportional to the differentiation of the input signal. We obtain a differentiator by replacing the input resistor in the inverting amplifier by a capacitor. This gives



$$v_o(t) = -RC\frac{d}{dt}v_i(t).$$

- Differentiator circuits are electronically unstable because any electrical noise within the circuit is exaggerated by the differentiator. For this reason, the differentiator circuit above is not as useful and popular as the integrator. It is seldom used in practice.

CHAPTER 7

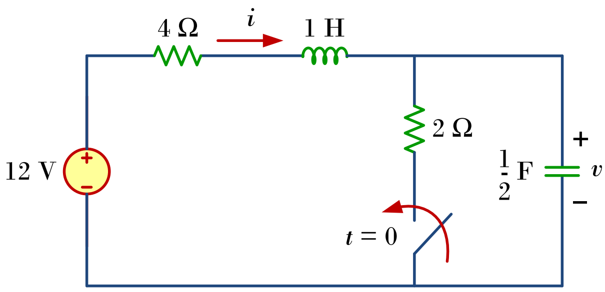
Sinusoids and Phasors

Recall that, for capacitors and inductors, the branch variables (current values and voltage values) are related by differential equations. Normally, to analyze a circuit containing capacitor and/or inductor, we need to solve some differential equations. The analysis can be greatly simplifies when the circuit is driven (or excited) by a source (or sources) that is sinusoidal. Such assumption will be the main focus of this chapter.

7.1. Prelude to Second-Order Circuits

The next example demonstrates the complication normally involved when analyzing a circuit containing capacitor and inductor. This example and the analysis presented is not the main focus of this chapter.

EXAMPLE 7.1.1. The switch in the figure below has been open for a long time. It is closed at $t = 0$.



(a) Find $v(0)$ and $\frac{dv}{dt}(0)$.
(b) Find $v(t)$ for $t > 0$.
(c) Find $v(\infty)$ and $\frac{dv}{dt}(\infty)$.
(d) Find $v(t)$ for $t > 0$ when the source is $v_s(t) = \begin{cases} 12, & t < 0, \\ 12\cos(t), & t \geq 0. \end{cases}$

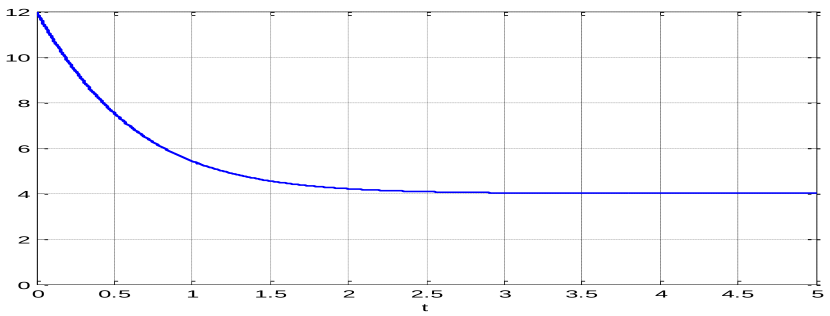
From MATLAB,

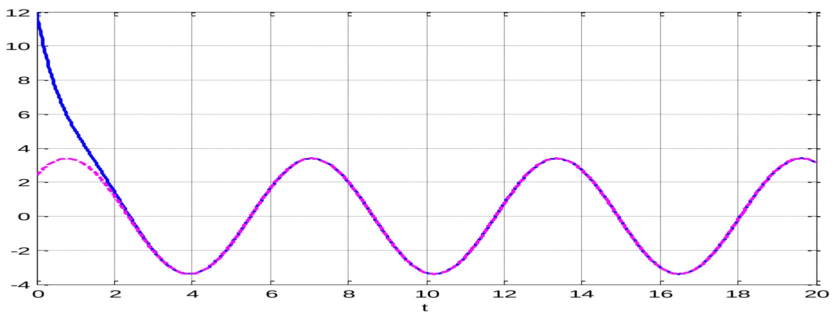
```
v = dsolve('D2v + 5*Dv + 6*v = 24','v(0) = 12','Dv(0) = -12')
```

gives $v(t) = 4 + 12e^{-2t} - 4e^{-3t}$. Similarly,

```
v = dsolve('D2v + 5*Dv + 6*v = 2*12*cos(t)','v(0) = 12','Dv(0) = -12','t')
```

gives $v(t) = \frac{72}{5}e^{-2t} - \frac{24}{5}e^{-3t} + \frac{12}{5}\cos(t) + \frac{12}{5}\sin(t)$.





7.2. Sinusoids

DEFINITION 7.2.1. Some terminology:

(a) A **sinusoid** is a signal (, e.g. voltage or current) that has the form of the sine or cosine function.
   - Turn out that you can express them all under the same notation using only cosine (or only sine) function.
     - We will use cosine.

(b) A sinusoidal current is referred to as **alternating current (AC)**.

(c) We use the term **AC source** for any device that supplies a sinusoidally varying voltage (potential difference) or current.

(d) Circuits driven by sinusoidal current or voltage sources are called **AC circuits**.

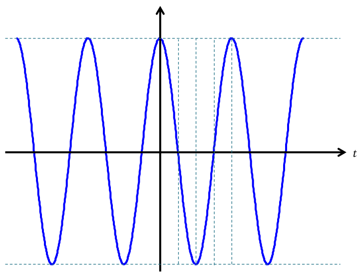
7.2.2. Consider the sinusoidal signal (in cosine form)

$$x(t) = X_m \cos(\omega t + \phi) = X_m \cos(2\pi f t + \phi),$$

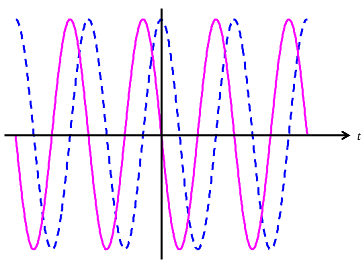
where

X_m: the amplitude of the sinusoid,
ω: the angular frequency in radians/s (or rad/s),
ϕ: the phase.

- First, we consider the case when $\phi = 0$:



- When $\phi \neq 0$, we shift the graph of $X_m \cos(\omega t)$ to the ***left*** "by ϕ".



7.2.3. The **period** (the time of one complete cycle) of the sinusoid is

$$T = \frac{2\pi}{\omega}.$$

The unit of the period is in second if the angular frequency unit is in radian per second.

The **frequency** f (the number of cycles per second or hertz (Hz)) is the reciprocal of this quantity, i.e.,

$$f = \frac{1}{T}.$$

7.2.4. ***Standard form*** for sinusoid: In this class, when you are asked to find the sinusoid representation of a signal, make sure that your answer is in the form

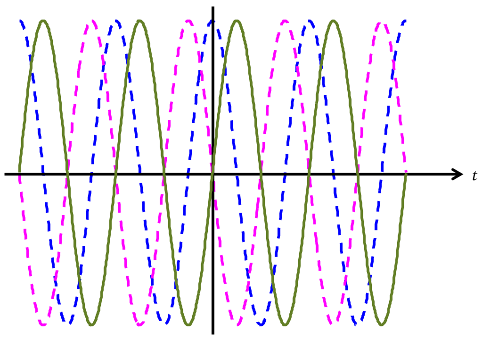
$$x(t) = X_m \cos(\omega t + \phi) = X_m \cos(2\pi f t + \phi),$$

where X_m is nonnegative and ϕ is between -180° and $+180^\circ$.

7.2.5. Conversions to standard form

- When the signal is given in the sine form, it can be converted into its cosine form via the identity

$$\sin(x) = \cos(x - 90^\circ).$$



In particular,

$$X_m \sin(\omega t + \phi) = X_m \cos(\omega t + \phi - 90^\circ).$$

- X_m is always non-negative. We can avoid having the negative sign by the following conversion:

$$-\cos(x) = \cos(x \pm 180^\circ).$$

In particular,

$$-A\cos(\omega t+\phi) = A\cos(2\pi ft+\phi \pm 180°).$$

Note that usually you do not have the choice between +180° or −180°. The one that you need to use is the one that makes $\phi \pm 180°$ falls somewhere between −180° and +180°.

7.2.6. For any[1] linear AC circuit, the "steady-state" voltage and current are sinusoidal with the same frequency as the driving source(s).

- Although all the voltage and current are sinusoidal, their amplitudes and phases can be different.
  - These can be found by the technique discussed in this chapter.

7.3. Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions. The tradeoff is that phasors are complex-valued.

7.3.1. The idea of phasor representation is based on Euler's identity:

$$e^{j\phi} = \cos\phi + j\sin\phi,$$

From the identity, we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$:

$$\cos\phi = \mathrm{Re}\left\{e^{j\phi}\right\}, \quad \sin\phi = \mathrm{Im}\left\{e^{j\phi}\right\},$$

where Re and Im stand for "the real part of" and "the imaginary part of" $e^{j\phi}$.

Definition 7.3.2. A ***phasor*** is a complex number that represents the amplitude and phase of a sinusoid. Given a sinusoid $x(t) = X_m\cos(\omega t+\phi)$, then

$$x(t) = X_m\cos(\omega t+\phi) = \mathrm{Re}\left\{X_m e^{j(\omega t+\phi)}\right\} = \mathrm{Re}\left\{X_m e^{j\phi}\cdot e^{j\omega t}\right\} = \mathrm{Re}\left\{\mathbf{X}e^{j\omega t}\right\},$$

where

$$\mathbf{X} = X_m e^{j\phi} = X_m\angle\phi.$$

The complex number $\mathbf{X}$ is called the ***phasor representation*** of the sinusoid $v(t)$. Notice that a phasor captures information about amplitude and phase of the corresponding sinusoid.

[1]When there are multiple sources, we assume that all sources are at the same frequency.

7.3.3. Whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is **implicit**. It is therefore important, when dealing with phasors, to keep in mind the frequency f (or the angular frequency ω) of the phasor.

7.3.4. Given a phasor $\mathbf{X}$, to obtain the time-domain sinusoid corresponding to a given phasor, there are two important routes.

(a) Simply write down the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor.
(b) Multiply the phasor by the time factor $e^{j\omega t}$ and take the real part.

7.3.5. Any complex number z (including any phasor) can be equivalently represented in three forms.

(a) Rectangular form: $z = x + jy$.
(b) Polar form: $z = r\angle\phi$.
(c) Exponential form: $z = re^{j\phi}$

where the relations between them are

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\frac{y}{x} \pm 180^\circ.$$

$$x = r\cos\phi, \quad y = r\sin\phi.$$

Note that for ϕ, the choice of using $+180^\circ$ or -180° in the formula is determined by the actual quadrant in which the complex number lies.

As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. In this class, we focus on polar form.

7.3.6. ***Summary***: By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$x(t) = X_m\cos(\omega t + \phi) \Leftrightarrow \mathbf{X} = X_m\angle\phi.$$

Time domain representation $\Leftrightarrow$ Phasor domain representation

DEFINITION 7.3.7. ***Standard form*** for phasor: In this class, when you are asked to find the phasor representation of a signal, make sure that your answer is a complex number in polar form, i.e. $r\angle\phi$ where r is nonnegative and ϕ is between -180° and $+180^\circ$.

EXAMPLE 7.3.8. Transform these sinusoids to phasors:
(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

EXAMPLE 7.3.9. Find the sinusoids represented by these phasors:
(a) $\mathbf{I} = -3 + j4$ A

(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

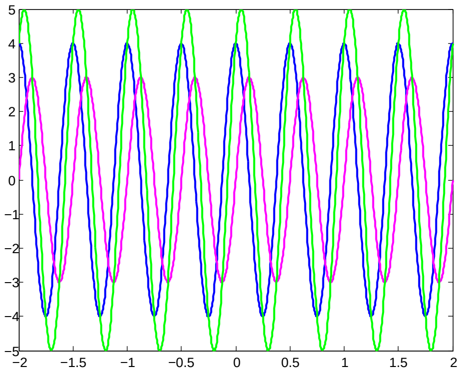
7.3.10. The differences between $x(t)$ and $\mathbf{X}$ should be emphasized:
(a) $x(t)$ is the instantaneous or time-domain representation, while $\mathbf{X}$ is the frequency or phasor-domain representation.
(b) $x(t)$ is time dependent, while $\mathbf{X}$ is not.
(c) $x(t)$ is always real with no complex term, while $\mathbf{X}$ is generally complex.

7.3.11. Adding sinusoids of the *same frequency* is equivalent to adding their corresponding phasors. To see this,

$$\begin{aligned} A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2) &= \operatorname{Re}\left\{\mathbf{A}_1 e^{j\omega t}\right\} + \operatorname{Re}\left\{\mathbf{A}_2 e^{j\omega t}\right\} \\ &= \operatorname{Re}\left\{(\mathbf{A}_1 + \mathbf{A}_2)\, e^{j\omega t}\right\}. \end{aligned}$$

Because $\mathbf{A}_1 + \mathbf{A}_2$ is just another complex number, we can conclude a (surprising) fact: adding two sinusoids of the same frequency gives another sinusoids.

EXAMPLE 7.3.12. $x(t) = 4\cos(2t) + 3\sin(2t)$



7.3.13. Properties involving differentiation and integration:

(a) **Differentiating** a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$. In other words,

$$\frac{dx(t)}{dt} \Leftrightarrow j\omega\mathbf{X}.$$

To see this, suppose $x(t) = X_m \cos(\omega t + \phi)$. Then,

$$\frac{dx}{dt}(t) = -\omega X_m \sin(\omega t + \phi) = \omega X_m \cos(\omega t + \phi - 90^\circ + 180^\circ)$$
$$= \operatorname{Re}\left\{\omega X_m e^{j\phi} e^{j90^\circ} \cdot e^{j\omega t}\right\} = \operatorname{Re}\left\{j\omega\mathbf{X}e^{j\omega t}\right\}$$

Alternatively, express $v(t)$ as

$$x(t) = \operatorname{Re}\left\{X_m e^{j(\omega t+\phi)}\right\}.$$

Then,

$$\frac{d}{dt}x(t) = \operatorname{Re}\left\{X_m j\omega e^{j(\omega t+\phi)}\right\}.$$

(b) **Integrating** a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$. In other words,

$$\int x(t)dt \Leftrightarrow \frac{\mathbf{X}}{j\omega}.$$

EXAMPLE 7.3.14. Find the voltage $v(t)$ in a circuit described by the intergrodifferential equation

$$2\frac{dv}{dt} + 5v + 10\int vdt = 50\cos(5t - 30^\circ)$$

using the phasor approach.

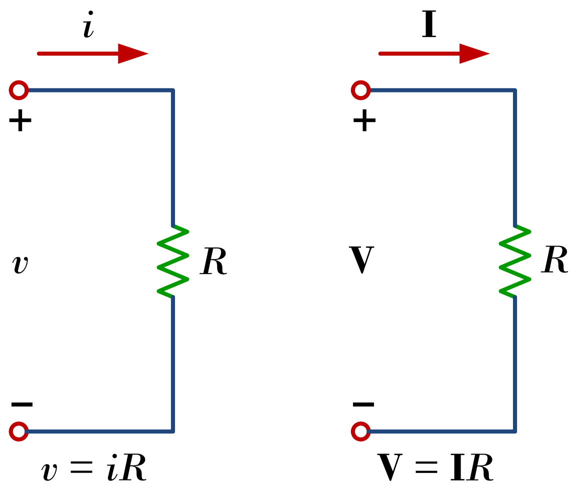
7.4. Phasor relationships for circuit elements

7.4.1. Resistor R: If the current through a resistor R is

$$i(t) = I_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{I} = I_m \angle \phi,$$

the voltage across it is given by

$$v(t) = i(t)R = RI_m \cos(\omega t + \phi).$$



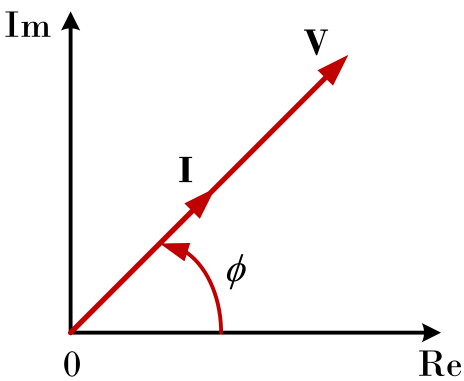
The phasor of the voltage is

$$\mathbf{V} = RI_m \angle \phi.$$

Hence,

$$\mathbf{V} = \mathbf{I}R.$$

We note that voltage and current are **in phase** and that the voltage-current relation for the resistor in the phasor domain continues to be Ohms law, as in the time domain.

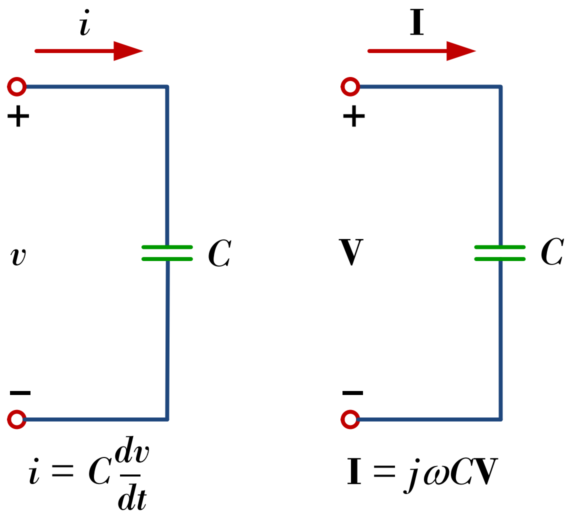


7.4.2. Capacitor C: If the voltage across a capacitor C is

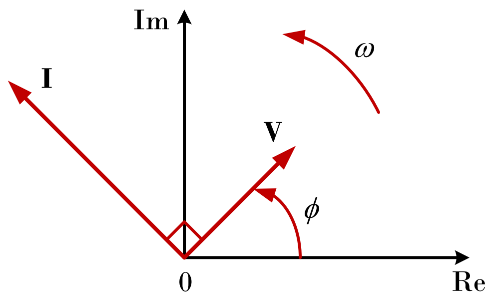
$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m \angle \phi,$$

the current through it is given by

$$i(t) = C\frac{dv(t)}{dt} \Leftrightarrow \mathbf{I} = j\omega C\mathbf{V} = \omega C V_m \angle(\phi + 90°).$$



The voltage and current are 90° out of phase. Specifically, the current leads the voltage by 90°.



- Mnemonic: CIVIL

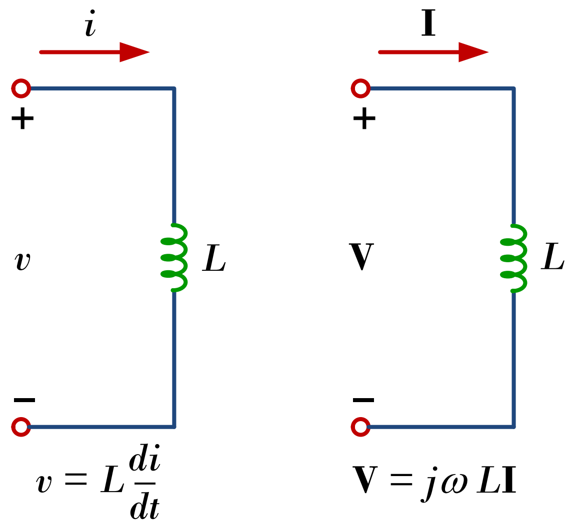
  In a Capacitive (C) circuit, I leads V. In an inductive (L) circuit, V leads I.

7.4.3. Inductor L: If the current through an inductor L is

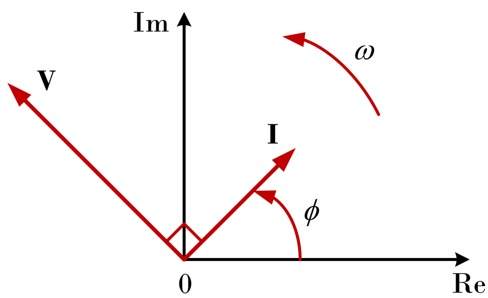
$$i(t) = I_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{I} = I_m \angle \phi,$$

the voltage across it is given by

$$v(t) = L\frac{di(t)}{dt} \Leftrightarrow \mathbf{V} = j\omega L\mathbf{I} = \omega L I_m \angle(\phi + 90°).$$



The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°.



Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$

7.5. Impedance and Admittance

In the previous part, we obtained the voltage current relations for the three passive elements as

$$\mathbf{V} = \mathbf{I}R, \quad \mathbf{V} = j\omega L\mathbf{I}, \quad \mathbf{I} = j\omega C\mathbf{V}.$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor of current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \quad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \quad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}.$$

From these equations, we obtain Ohm's law in phasor form for any type of element as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{I}\mathbf{Z}.$$

Definition 7.5.1. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage $\mathbf{V}$ to the phasor current $\mathbf{I}$, measured in ohms (Ω).

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX = |\mathbf{Z}|\angle\theta,$$

with

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\frac{X}{R}, \quad R = |\mathbf{Z}|\cos\theta, \quad X = |\mathbf{Z}|\sin\theta.$$

$R = \operatorname{Re}\{\mathbf{Z}\}$ is called the **resistance** and $X = \operatorname{Im}\{\mathbf{Z}\}$ is called the **reactance**.

The reactance X may be positive or negative. We say that the impedance is **inductive** when X is positive or **capacitive** when X is negative.

Definition 7.5.2. The **admittance** ($\mathbf{Y}$) is the reciprocal of impedance, measured in Siemens (S). The admittance of an element(or a circuit) is the ratio of the phasor current through it to phasor voltage across it, or

$$\mathbf{Y} = \frac{\mathbf{1}}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}.$$

7.5.3. **Kirchhoff's laws (KCL and KVL) hold in the phasor form**.

To see this, suppose $v_1, v_2, \ldots, v_n$ are the voltages around a closed loop, then

$$v_1 + v_2 + \cdots + v_n = 0.$$

If each voltage v_i is a sinusoid, i.e.

$$v_i = V_{mi}\cos(\omega t + \phi_i) = \text{Re}\left\{\mathbf{V}_i e^{j\omega t}\right\}$$

with phasor $\mathbf{V}_i = V_{mi}\angle\phi_i = V_{mi}e^{j\phi_i}$, then

$$\text{Re}\left\{\left(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n\right)e^{j\omega t}\right\} = 0,$$

which must be true for all time t. To satisfy this, we need

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0.$$

Hence, KVL holds for phasors.

Similarly, we can show that KCL holds in the frequency domain, i.e., if the currents $i_1, i_2, \ldots, i_n$ are the currents entering or leaving a closed surface at time t, then

$$i_1 + i_2 + \cdots + i_n = 0.$$

If the currents are sinusoids and $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ are their phasor forms, then

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0.$$

7.5.4. **Major Implication:** Since Ohm's Law and Kirchoff's Laws hold in phasor domain, **all resistance combination formulas, volatge and current divider formulas, analysis methods** (nodal and mesh analysis) **and circuit theorems** (linearity, superposition, source transformation, and Thevenin's and Norton's equivalent circuits) that we have previously studied for dc circuits **apply to ac circuits** !!!

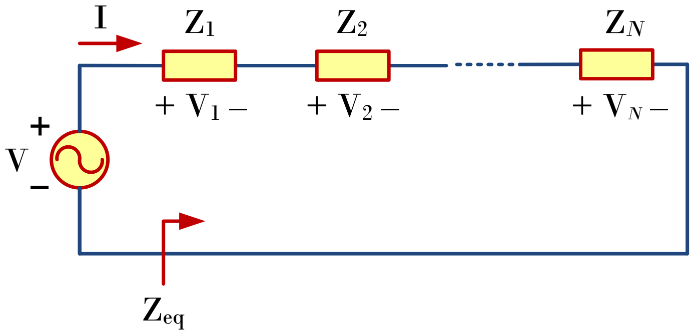
Just think of impedance as a complex-valued resistance!!

The three-step analysis in the next chapter is based on this insight.

In addition, our ac circuits can now effortlessly include capacitors and inductors which can be considered as impedances whose values depend on the frequency ω of the ac sources!!

7.6. Impedance Combinations

Consider N series-connected impedances as shown below.



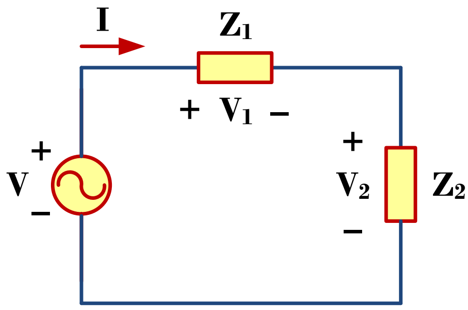
The same current $\mathbf{I}$ flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N.$$

In particular, if $N = 2$, the current through the impedance is



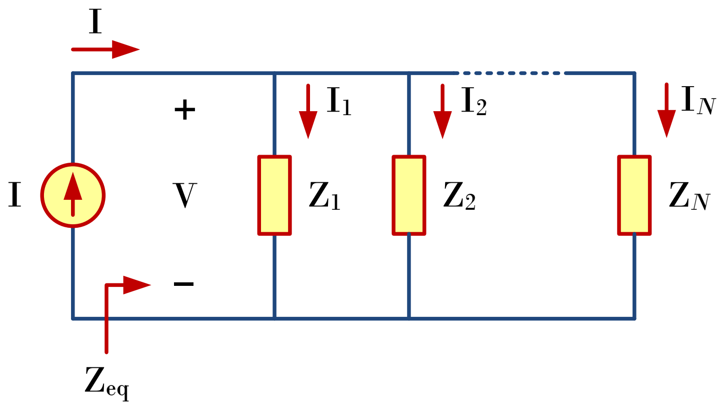
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

Because $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$,

$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \quad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

which is the **voltage-division** relationship.

Now, consider N parallel-connected impedances as shown below.



The voltage across each impedance is the same. Applying KCL at the top node gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right).$$

The equivalent impedance $\mathbf{Z}_{eq}$ can be found from

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}.$$

When $N = 2$,

$$\mathbf{Z}_{eq} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

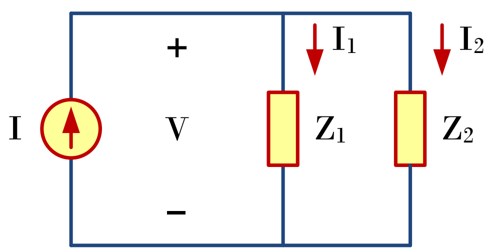
Because

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2,$$

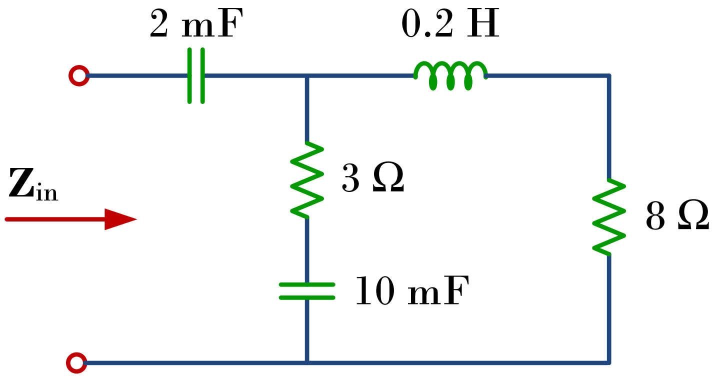
we have

$$\mathbf{I_1} = \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{I}, \quad \mathbf{I_2} = \frac{\mathbf{Z_1}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{I}$$

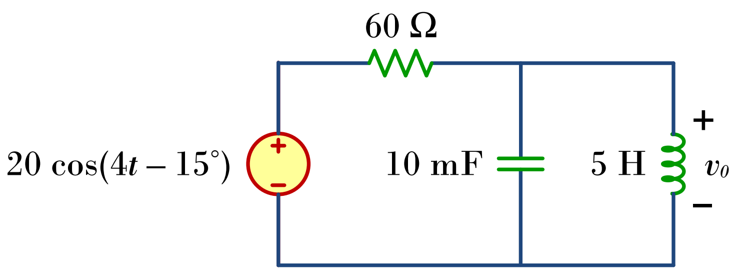
which is the **current-division** principle.



EXAMPLE 7.6.1. Find the input impedance of the circuit below. Assume that the circuit operates at $\omega = 50$ rad/s.



EXAMPLE 7.6.2. Determine $v_o(t)$ in the circuit below.



CHAPTER 8

Sinusoidal Steady State Analysis

8.1. General Approach

In the previous chapter, we have learned that the steady-state response of a circuit to sinusoidal inputs can be obtained by using phasors. In this chapter, we present many examples in which nodal analysis, mesh analysis, Thevenin's theorem, superposition, and source transformations are applied in analyzing ac circuits.

8.1.1. **Steps to analyze ac circuits, using phasor domain:**

Step 1. Transform the circuit to the phasor or frequency domain.
- Not necessary if the problem is specified in the frequency domain.

Step 2. Solve the problem using circuit techniques (e.g., nodal analysis, mesh analysis, Thevenin's theorem, superposition, or source transformations)
- The analysis is performed in the same manner as dc circuit analysis except that complex numbers are involved.

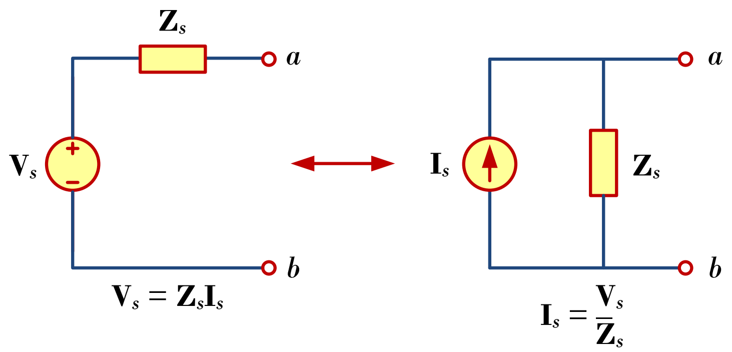
Step 3. Transform the resulting phasor back to the time domain.

8.1.2. ac circuits are linear (they are just composed of sources and impedances)

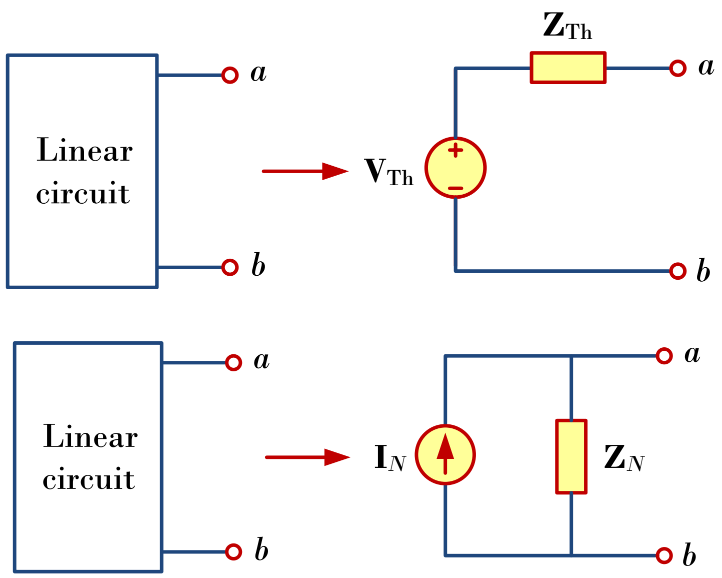
8.1.3. The **superposition theorem** applies to ac circuits the same way it applies to dc circuits. This is the case when all the sources in the circuit operate at the same frequency. If they are operating at different frequency, see Section 8.2.

8.1.4. **Source transformation**:

$$\mathbf{V_s} = \mathbf{Z_s}\mathbf{I_s}, \quad \mathbf{I_s} = \frac{\mathbf{V_s}}{\mathbf{Z_s}}.$$

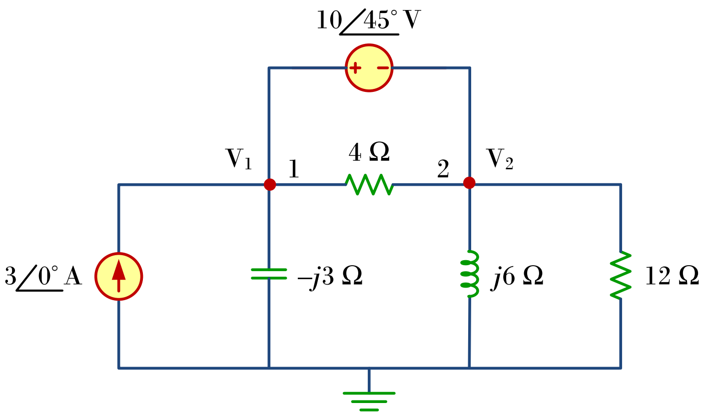


8.1.5. **Thevenin and Norton Equivalent circuits**:

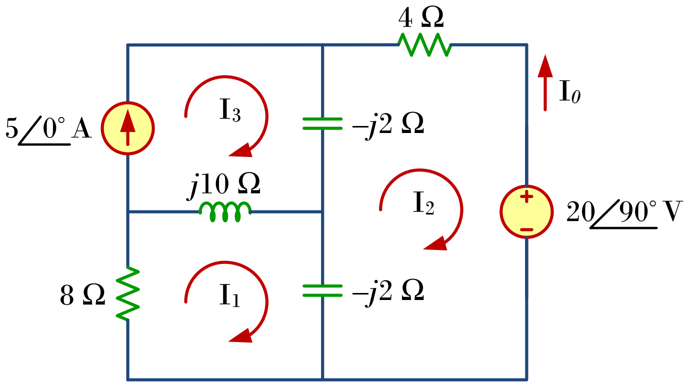


$$\mathbf{V_{Th}} = \mathbf{Z_N}\mathbf{I_N}, \quad \mathbf{Z_{Th}} = \mathbf{Z_N}$$

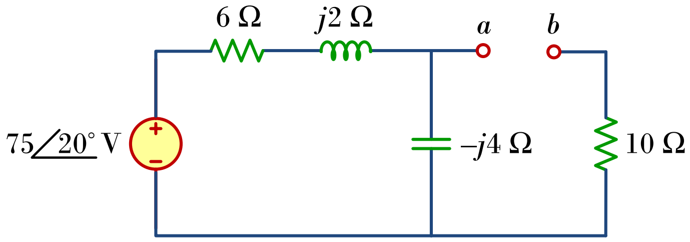
Example 8.1.6. Compute $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit below using nodal analysis.



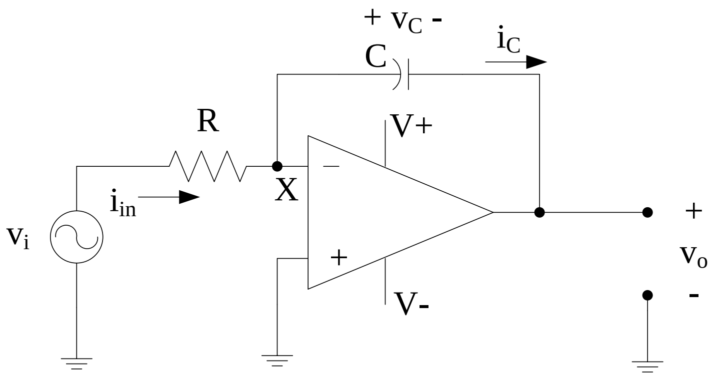
Example 8.1.7. Determine current $\mathbf{I}_o$ in the circuit below using mesh analysis.



EXAMPLE 8.1.8. Find the Thevenin equivalent at terminals a-b of the circuit below.



EXAMPLE 8.1.9. **Op Amp AC Circuits**: Find the (closed-loop) gain of the circuit below.



8.2. Circuit With Multiple Sources Operating At Different Frequencies

A special care is needed if the circuit has multiple sources operating at different frequencies. In which case, one must add the responses due to the individual frequencies in the time domain. In other words, the superposition still works but

(a) We must have a different frequency-domain circuit for each frequency.
(b) The total response must be obtained by adding the individual response in the time domain.

8.2.1. Since the impedance depend on frequency, it is incorrect to try to add the responses in the phasor or frequency domain. To see this note that the exponential factor $e^{j\omega t}$ is implicit in sinusoidal analysis, and that factor would change for every angular frequency ω. In particular, although

$$\sum_i V_{mi}\cos(\omega t+\phi_i)=\sum_i \operatorname{Re}\left\{\mathbf{V}_i e^{j\omega t}\right\}=\operatorname{Re}\left\{\left(\sum_i \mathbf{V}_i\right)e^{j\omega t}\right\},$$

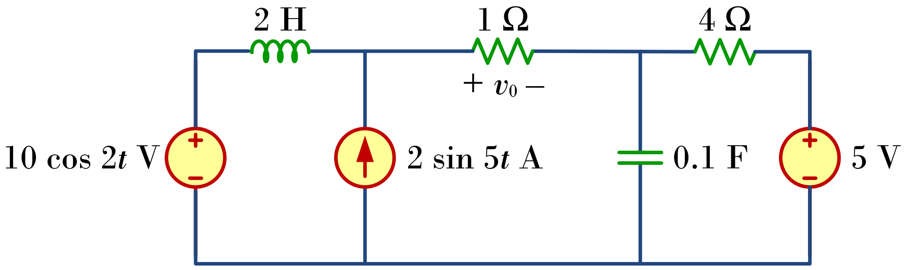
when we allow ω to be different for each sinusoid, generally

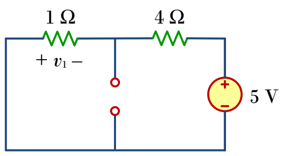
$$\sum_i V_{mi}\cos(\omega_i t+\phi_i)=\sum_i \operatorname{Re}\left\{\mathbf{V}_i e^{j\omega_i t}\right\}\neq\operatorname{Re}\left\{\left(\sum_i \mathbf{V}_i\right)e^{j\omega_i t}\right\}.$$

Therefore, it does not make sense to add responses at different frequencies in the phasor domain.

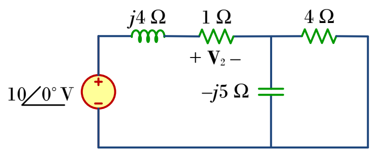
8.2.2. The Thevenin or Norton equivalent circuit (if needed) must be determined at each frequency and we have one equivalent circuit for each frequency.

EXAMPLE 8.2.3. Find v_o in the circuit below using the superposition theorem.

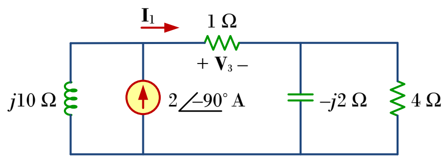




(a)



(b)



(c)